Arbitrary Prices in Generalized Bertrand Competition

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Problem: Consider a Generalized Bertrand Competition game. There are N firms indexed by i, each with cost c_i. They simultaneously announce prices p_i and realize demand $q_i(p_i, p_{-i})$, which is a function both of their own price, and other firms' prices. Their profits are

$$\pi_i(p_i, p_{-i}) = q_i(p_i, p_{-i})(p_i - c_i)$$

Construct demand functions q_i such that it is a Nash Equilibrium for each firm to announce the same price $p_i = p$ **for any price** $p > c_i \forall i$.

The demand functions should satisfy the Law of Demand:

$$\frac{\partial q_i}{\partial p_i} < 0$$

And the firms' products should be substitutes:

$$\frac{\partial q_i}{\partial p_{-i}} > 0$$

Solution: Take the first-order condition with respect to p_i:

$$\frac{\partial \pi_i}{\partial p_i} = q_i + \frac{\partial q_i}{\partial p_i}(p_i - c_i) = 0$$

We seek an equilibrium where alll firms set the same price. So we will conjecture that this equation will be solved by $p_i = p$ when all other prices $p_j = p \forall j \neq i$. To do this, conjecture that the optimal p_i is some function of p_{-i} such that the function evaluates to p if all arguments (the other prices) are p. Many functions will work here, but the most natural is simply the average function:

$$p_i = \overline{p}_{-i} \equiv \frac{1}{N-1}\sum_{j\neq i} p_j$$

Substitute this into the F.O.C.:

$$\begin{aligned} 0 &= q_i + \frac{\partial q_i}{\partial p_i}\left(\overline{p}_{-i} - c_i\right) \\ \frac{\partial q_i}{\partial p_i} &= -\frac{q_i}{\left(\overline{p}_{-i} - c_i\right)} \end{aligned}$$

This is a first-order O.D.E. with a simple solution

$$q_i\left(p_i, p_{-i}\right) = \exp\left(-\frac{p_i}{\overline{p}_{-i} - c_i}\right)$$

This is the solution to the problem. The remaining work is simply to formally validate what may seem obvious by inspecting the equation.

The function is maximized, not minimized, at $\overline{p}_{-i}$, as it is continuous and fulfills the Second-Order Condition:

$$\begin{aligned} \frac{\partial^2 \pi_i}{\partial p_i^2} &= \frac{\partial q_i}{\partial p_i} + \frac{\partial q_i}{\partial p_i} + \frac{\partial^2 q_i}{\partial p_i}\left(p_i - c_i\right) \\ &= -\frac{2}{\overline{p}_{-i} - c_i}\exp\left(-\frac{p_i}{\overline{p}_{-i} - c_i}\right) + \left(\frac{1}{\overline{p}_{-i} - c_i}\right)^2 \exp\left(-\frac{p_i}{\overline{p}_{-i} - c_i}\right)\left(p_i - c_i\right) \\ &= \left(\left(p_i - c_i\right)\left(\frac{1}{\overline{p}_{-i} - c_i}\right)^2 - \frac{2}{\overline{p}_{-i} - c_i}\right)\exp\left(-\frac{p_i}{\overline{p}_{-i} - c_i}\right) \\ &= \left(\frac{1}{p_i - c_i} - \frac{2}{p_i - c_i}\right)\exp\left(-\frac{p_i}{p_i - c_i}\right) \\ &= -\frac{1}{p_i - c_i}\exp\left(-\frac{p_i}{p_i - c_i}\right) < 0 \end{aligned}$$

For the fourth line above, recall that $\overline{p}_{-i} = p_i$.

It satisfies the Law of Demand:

$$\frac{\partial q_i}{\partial p_i} = -\frac{1}{\overline{p}_{-i} - c_i}\exp\left(-\frac{p_i}{\overline{p}_{-i} - c_i}\right) < 0$$

And the products are substitutes:

$$
\begin{aligned}
\frac{\partial q_i}{\partial p_{j\neq i}} &= \frac{\partial}{\partial p_j} \exp\left(-\frac{p_i}{\frac{1}{N-1}p_j + \frac{N-2}{N-1}p_i - c_i}\right) \\
&= -\exp\left(-\frac{p_i}{\frac{1}{N-1}p_j + \frac{N-2}{N-1}p_i - c_i}\right) \frac{\partial}{\partial p_j}\left(\frac{p_i}{\frac{1}{N-1}p_j + \frac{N-2}{N-1}p_i - c_i}\right) \\
&= -\exp\left(-\frac{p_i}{\frac{1}{N-1}p_j + \frac{N-2}{N-1}p_i - c_i}\right)\left(\frac{-\frac{1}{N-1}p_i}{\left(\frac{1}{N-1}p_j + \frac{N-2}{N-1}p_i - c_i\right)^2}\right) \\
&= (N-1)\,\frac{p_i}{\left(\overline{p}_i - c_i\right)^2} \exp\left(-\frac{p_i}{\overline{p_i} - c_i}\right) > 0
\end{aligned}
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Q.E.D.