Cox-Ingersoll-Ross Term Structure Model

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Question

Consider an economy with a single state variable x_t, whose process is

$$x_{t+1} = \mu + \phi x_t + \sigma x_t^{1/2} \varepsilon_{t+1}$$

The SDF evolves as

$$m_{t+1} = -x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t - \left(\frac{\lambda}{\sigma}\right) x_t^{1/2} \varepsilon_{t+1}$$

$\varepsilon_{t+1} \sim N(0,1)$

a. Calculate the one-period (riskless) log bond yield.

b. Write an equation relating the log price of an n-period bond at time t to the future log price of an $n-1$-period bond, $p_{n-1,t+1}$. (Hint: the state variable x_t and the future SDF m_{t+1} show up in the answer).

c. Conjecture that the price of a bond is linear in the state variable:

$$p_{nt} = A_n + B_n x_t$$

Write recursive formulae for $A_n$and B_n. Be sure to include the initial condition.

d. What is the term premium on an n-period bond?

e. What is the Sharpe ratio of an n-period bond?

f. What empirical behavior of interest rates can be explained by this model, as opposed to the simplest completely affine homoskedastic term structure model that one could write? What predictions of this model are counterfactual?

Solutions

a. The price of a risk-free bond is the expectation of the SDF:

$$\begin{aligned}
P_{1,t} &= E\left[M_{t+1}\right] \\
p_{1,t} &= E\left[m_{t+1}\right] + \frac{1}{2} Var\left[m_{t+1}\right] \\
&= \left(-x_t - \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t\right) + \frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t \\
&= -x_t \\
y_{1,t} = -p_{1,t} &= x_t
\end{aligned}$$

Recall that this is the same result as in the homoskedastic case.

b. No-arbitrage tells us

$$\begin{aligned}
1 &= E_t\left[M_{t+1} R_{t+1}\right] \\
1 &= E_t\left[M_{t+1} \frac{P_{t+1}}{P_t}\right] \\
P_t &= E_t\left[M_{t+1} P_{t+1}\right]
\end{aligned}$$

Recall that an n-period bond today is an $n-1$-period bond tomorrow,

$$P_{n,t} = E_t\left[M_{t+1} P_{n-1,t+1}\right]$$

Now take logs and apply Jensen's inequality,

$$\begin{aligned}
p_{n,t} &= E_t\left[m_{t+1} + p_{n-1,t+1}\right] + \frac{1}{2} Var\left[m_{t+1} + p_{n-1,t+1}\right] \\
&= \underbrace{E_t\left[m_{t+1}\right] + \frac{1}{2} Var\left[m_{t+1}\right]}_{-x_t} + E_t\left[p_{n-1,t+1}\right] + \frac{1}{2} Var\left[p_{n-1,t+1}\right] + Cov\left[m_{t+1}, p_{n-1,t+1}\right]
\end{aligned}$$

c. Plug in the conjecture into the equation above,

$$A_n + B_n x_t = -x_t + A_{n-1} + B_{n-1}\left(\mu + \phi x_t\right) + \frac{1}{2}\left(B_{n-1}\sigma\right)^2 x_t - B_{n-1}\lambda x_t$$

Now, use variation of parameters to match coefficient on the units and on the x_t:

$$A_n = A_{n-1} + B_{n-1}\mu$$

$$B_n = -1 + B_{n-1}\phi + \frac{\left(B_{n-1}\sigma\right)^2}{2} - B_{n-1}\lambda$$

These are the recursions. The initial conditions are that $p_{1,t} = -x_t$, so

$$A_1 = 0$$

$$B_1 = -1$$

d. The term premium is

$$E\left[r_{n,t+1}\right] + \frac{1}{2}Var\left[r_{n,t+1}\right] - y_{1,t} = -Cov\left(p_{n-1,t+1}, m_{t+1}\right)$$

To prove this to yourself,

$$E\left[MR\right] = 1$$

$$\underbrace{E\left[m\right] + \frac{1}{2}Var\left[m\right]}_{-rf} + E\left[r\right] + \frac{1}{2}Var\left[r\right] + Cov\left[m, r\right] = 0$$

$$E\left[r\right] + \frac{1}{2}Var\left[r\right] - rf = -Cov\left[m, r\right]$$

So with this fact, the term premium falls right out of the covariance,

$$
\begin{aligned}
E\left[r_{n,t+1}\right]+\frac{1}{2}Var\left[r_{n,t+1}\right]-y_{1,t} &= -Cov_t\left(r_{n-1,t+1},m_{t+1}\right) \\
&= -Cov_t\left(p_{n-1,t+1}+p_{n,t},m_{t+1}\right) \\
&= -Cov_t\left(A_{n-1}+B_{n-1}\left(\mu+\phi x_t+\sigma x_t^{1/2}\varepsilon\right),-x_t-\frac{1}{2}\left(\frac{\lambda}{\sigma}\right)^2 x_t-\left(\frac{\lambda}{\sigma}\right)x_t^{1/2}\varepsilon_{t+1}\right) \\
&= -Cov_t\left(B_{n-1}\sigma x_t^{1/2}\varepsilon,-\left(\frac{\lambda}{\sigma}\right)x_t^{1/2}\varepsilon_{t+1}\right) \\
&= B_{n-1}\lambda x_t
\end{aligned}
$$

e. The Sharpe ratio is

$$
S_n=\frac{ERP}{Var\left[p_{n-1,t+1}\right]^{1/2}}=\frac{B_{n-1}\lambda x_t}{-B_{n-1}\sigma x_t^{1/2}}=-\left(\frac{\lambda}{\sigma}\right)x_t^{1/2}
$$

f. This model can rationalize the observed empirical pattern of time-varying risk premia.

Counterfactually, this model predicts that the term premium (increasing in x_t) should be positively correlated with the short rate x_t. In reality, we see larger term premiums when the short rate is lower.

Also counterfactually, this model predicts that a single factor can explain the term structure of interest rates. We know there are at least 3 factors for level, slope, and curvature (and potentially 4, depending on if you think the Cochrane-Piazzesi factor is real).